The limit distribution of inhomogeneous Markov processes and Kolmogorov's problem

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§1.1 Limit distribution of homogeneous Markov processes

Limit distribution

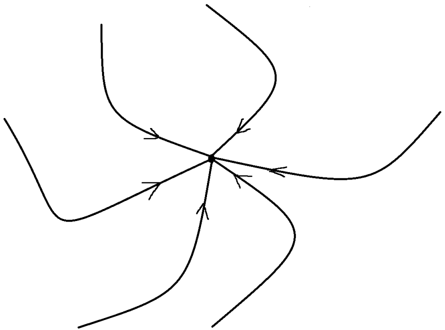
Theorem (Kolmogorov, Doeblin, Doob, Lévy, Chung, Harris, ...)

Let $X = \{X(t); t \geq 0\}$ be a Markov process. If X is Harris recurrent, then X admits a unique stationary distribution, which is the limit distribution of X.

Remark

If we denote by $\mu(t)$ the distribution of $X(t)$, then it means $\mu(t) \to \mu_\infty$ as $t \to \infty$.

What it looks like?



Convergence to a fixed point
(homogeneous case)

§1.2 Inhomogeneous Markov processes

Inhomogeneous is more natural

- From Doob's classical monograph "Stochastic Processes", to Chung's 'authoritative' "Markov Chains with Stationary Transition Probabilities", and to Meyn-Tweedie's 'bible' "Markov Chains and Stochastic Stability", only homogeneous case is treated.
- But it is very natural and important to consider inhomogeneous case, which means the developing mechanism of the system we consider varies as the time evolves.
- Due to the lack of tools, there are very few results on the limit distribution of inhomogeneous case.

Our aim: limit dis. for inhomogeneous MP

Natural questions:

- How to describe asymptotic behaviors of inhomogeneous Markov processes?
- Does there exist similar counterpart to stationary distribution in homogeneous case?

Our observation: for inhomogeneous MP generated by nonautonomous SDE, the theory and methods in dynamical systems can be used to study their limit behaviors.

§1.3 Our works on limit behaviors of inhomogeneous Markov Processes

Related works

Consider the semilinear SDE

$$dx(t) = (Ax(t) + F(t, x(t)))dt + G(t, x(t))dW(t) \quad (*)$$

with A generating a C^0-semigroup, $F, G \in C(\mathbb{R} \times H, H)$ and H a Hilbert space.

- Periodic: Khasminskii (69, Monograph), Zhao and coauthors (09-), Chen-Han-Li-Yang (17, JDE), ...
- Almost periodic: Halanay (87, P. Conf.), Morozan-Tudor (89, SAA), Da Prato-Tudor (95, SAA), Arnold-Tudor (98, Sto. Sto. Rep.), Bezandry-Diagana (07, App. Ana.), ...

Remark

These works only consider the existence of P/AP solutions, but do not concern the limit behaviors.

Our works on limit distribution

- Fu-L. (10, PAMS): Introduced almost automorphic (AA) stochastic processes and proved that the law of the unique AA solution is the limit distribution
- Wang-L. (12, Non.), L.-Sun (14, JFA): Introduced Lévy noise perturbation, and investigated big jump impact (intuitively harmful) on the limit distribution (AP/AA solution)
- Cheban-L. (17, preprint; 18, in preparation): The law of general[1] recurrent solutions is the limit distribution

[1] Includes stationary, periodic, quasi-periodic, almost periodic, pseudo-periodic, almost automorphic, Levitan almost periodic, Birkhoff recurrent, almost recurrent, pseudo-recurrent, Poisson stable.

Averaging and limit distribution

Consider the SDE

$$dx(t) = (Ax(t) + F(\frac{t}{\epsilon}, x(t)))dt + G(\frac{t}{\epsilon}, x(t))dW(t) \qquad (1)$$

with A, F, G the same as above, and the averaged SDE

$$dx(t) = (Ax(t) + \overline{F}(x(t)))dt + \overline{G}(x(t))dW(t). \qquad (2)$$

- Cheban-L. (18, preprint): the unique recurrent solution φ_ϵ of (1) will converge to the unique stationary solution $\bar{\varphi}$ of (2):

  $$\lim_{\epsilon \to 0} \sup_{t \in \mathbb{R}} \mathbf{E}|\varphi_\epsilon(t) - \bar{\varphi}|^2 = 0,$$

  and hence in distribution sense.

Remark

Different from existing works, our work considers averaging on infinite interval (Bogolyubov 2nd Th.)

AP limit distribution by separation and Lyapunov method

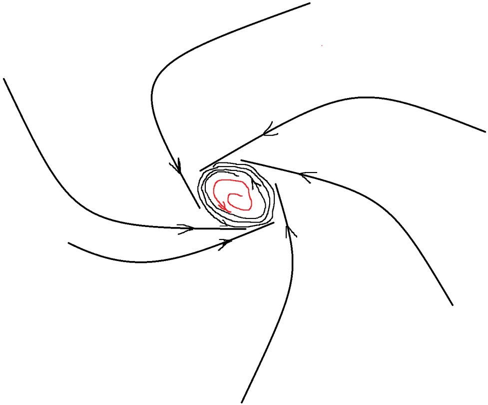
Consider the SDE on $\mathbb{R}^d$

$$\mathrm{d}X = f(t, X)\mathrm{d}t + g(t, X)\mathrm{d}W$$

with f, g AP in t.

- L.-Wang (16, JDE): boundedness + separation $\Longrightarrow$ AP solution
- Li-L.-Wang (19, DCDS-B): Lyapunov function + finite L^2-bounded solution $\Longrightarrow$ AP solution

Conclusion: an overview of inhomogeneous case



Convergence to a recurrent orbit
(inhomogeneous case)

§2.1 Kolmogorov's problem

Kolmogorov's problem

Stochastically perturbed ODE:

$$\mathrm{d}x = V(x)\mathrm{d}t + \sqrt{2\epsilon}\; G(x)\mathrm{d}W, \qquad x \in \mathbb{R}^n;$$

the associated stationary Fokker-Planck equation:

$$\epsilon\partial^2_{ij}(a^{ij}u) - \nabla\cdot(Vu) = 0, \qquad x \in \mathbb{R}^n$$

with $(a^{ij}) = GG^\top > 0$.

Kolmogorov's Problem[2] **(1950s):** the asymptotic behavior of stationary distribution μ_ϵ of the diffusion process as $\epsilon \to 0$.

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[2] Ya. Sinai (1989) calls it "famous problem". It will be seen below that this problem is closely related to stochastic stability.

§2.2 Existing works

Existing works

- Nevelson (1964): flows on the circle.
- Khasminskii (1963): flows on the 2-torus.
- Khasminskii (1963,1964): 2-D Hamiltonian systems.
- Freidlin-Wentzell (1970): the limit measures of μ_ϵ are supported on finite fixed points and periodic orbits.
- Kifer (1974), Young (1986): the limit of μ_ϵ is SRB measure for flow and diffeomorphism, respectively.

§2.3 Our work on Kolmogorov's problem

Joint with W. Huang, M. Ji, Y. Yi

(15, AoP; 15-16, JDDE I-III; 16, Phy. D.; 18, Phy. D.)

Lyapunov function

Consider the probability measure solution μ_ϵ to the above stationary F-P equation; typically it is an invariant measure of the SDE.

Definition

U is a Lyapunov function for the SDE or the F-P equation, if

$$LU = \epsilon a^{ij} \partial_{ij}^2 U + V \cdot \nabla U \leq -\gamma.$$

Remark

Note: if the SDE admits a Lyapunov function, then the unperturbed ODE admits the same one.

Theorem (Huang, Ji, L., Yi)

If the SDE admits a Lyapunov function, then

- the stationary measure μ_ϵ exists (even with degenerate diffusion);
- the family $\{\mu_\epsilon\}$ is tight, hence the set $\mathcal{M}$ of limit measures of μ_ϵ, as $\epsilon \searrow 0$, is nonempty;
- any limit measure $\mu \in \mathcal{M}$ is an invariant measure of the ODE, supported on the global attractor;
- (stochastic LaSalle invariance principle) if U is entirely weak Lyapunov, then any limit measure $\mu \in \mathcal{M}$ is supported on the set $\{x : V(x) \cdot \nabla U(x) = 0\}$.

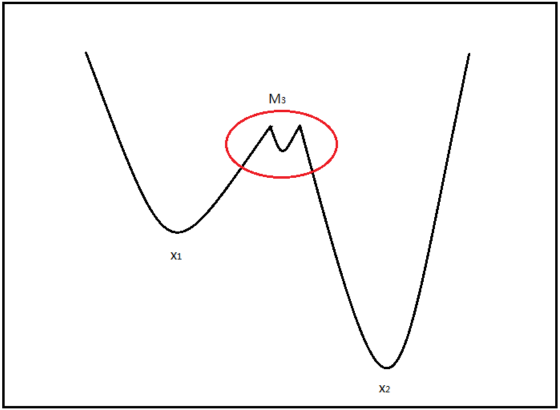
Remark

1. The set $\mathcal{M}$ depends essentially on G.
2. If $\mathcal{M} = \{\mu_0\}$ is a singleton set, then μ_0 is G-stable; if $\mathcal{M}$ not singleton, then none invariant measure of ODE can be G-stable.

Theorem (Huang, Ji, L., Yi)

If the ODE admits a Lyapunov function with bounded 2nd derivatives (*can be weakened*), then

- for a strong local attractor, there is G s.t. all the limit measures are supported on the attractor;
- for a strong local repeller, there is G s.t. all the limit measures are supported away the repeller;
- for a strong repelling equilibrium, for any typical G, all the limit measures are supported away the equilibrium.


M3
x1
x2

Connections with stochastic stability

Given finite # invariant sets $\{M_i\}_{i=1}^N$, as before, let $\mathcal{M}$ be the limit measures of μ_ϵ. Then

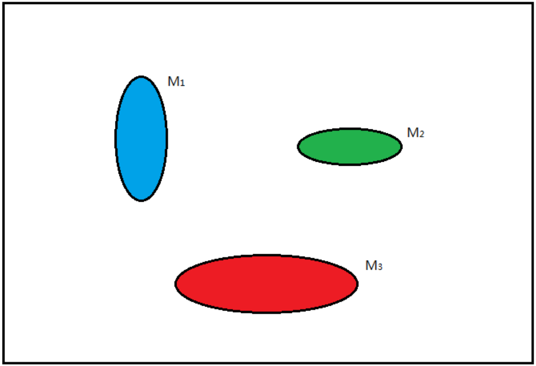
$$\bigcup_{\mu \in \mathcal{M}} \mathrm{supp}\mu \subset M_k \quad \Rightarrow \quad M_k \text{ is } G\text{-stable}$$

and others are G-negligible. If for $\epsilon \ll 1$,

$$\mu_\epsilon(M_i) < \mu_\epsilon(M_j) \quad \Rightarrow \quad M_j \text{ is more } G\text{-stable than } M_i.$$

Remark

1. In this way, we can discuss the (relative) stochastic stability of invariant sets under the perturbations $o(G\dot{W})$.
2. The stochastic stability of invariant measures is much more complicated than invariant sets, cf [16, Phy. D].



M1
M2
M3

THANK YOU!